Homework I: Point-Set Topology and Surfaces

due 28 Oct, 2010

1 Some Point-Set Problems

A We define the *half-infinite topology* on $\mathbb{R}$ to be generated by the set of all intervals $[a, \infty)$, for all $a \in \mathbb{R}$, along with the empty set.

(a) Prove that this is a topology:

(b) Is $\mathbb{R}$ with the half-infinite topology a Hausdorff space? (recall that a topological space $(\mathbb{X}, \mathcal{T})$ is Hausdorff if for each pair of distinct points $x, y \in \mathbb{X}$, there exists open sets $U, V \in \mathcal{T}$ such that $x \in U, y \in V$, and $U \cap V = \varnothing$):

(c) Is $\mathbb{R}$ with the half-infinite topology a connected space?

B Consider the unit-circle $S^1 = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$ as a subspace of $\mathbb{R}^2$ with the usual topology.

(a) Define $A = \{x \in S^1 \mid x_1 > 0\}$. Is A an open set in S^1? Is A a open set in $\mathbb{R}^2$?

(b) Define $B = \{x \in S^1 \mid x_1 \geq 0\}$. Is B a closed set in S^1? Is B a closed set in $\mathbb{R}^2$?

C Let $(\mathbb{X}, \mathcal{T})$ be an arbitrary topological space and pick some set $A \subset \mathbb{X}$. We define the *closure* of A to be the smallest closed set which contains A. We define the *interior* of A to be the largest open set which is contained in A. Now consider $\mathbb{R}$ with the usual topology and answer the following questions:

(a) Find the interior and the closure of the open interval $(0, 1)$:

(b) Find the interior and the closure of $\mathbb{Z}$, the set of all integers:

(c) Find the interior and the closure of $\mathbb{Q}$, the set of all rational numbers:

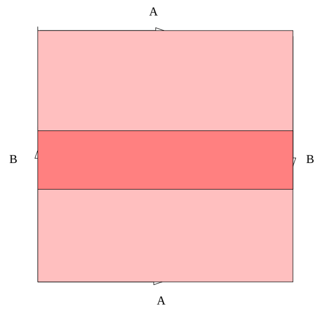

Figure 1: The Klein bottle: identify labeled edges with the orientation as shown by the arrows.

2 Manifold and Surface Problems

D Let $\mathbb{M}$ and $\mathbb{N}$ be two connected and compact 2-manifolds without boundary. Conjecture (and then prove) a formula which relates $\chi(\mathbb{M}\#\mathbb{N})$ to $\chi(\mathbb{M})$ and $\chi(\mathbb{N})$. (hint: if you have triangulated both manifolds, what's an easy combinatorial way to form the connected sum?)

E Consider the Klein bottle as drawn in Figure 1.

(a) What topological space is (up to homeomorphism) indicated by the darker strip in the figure? You may assume that the strip is perfectly centered:

(b) What topological space is obtained by removing the darker strip?

F Consider the unit sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^3 = 1\}$ as a subspace of $\mathbb{R}^3$ and let $n = (0, 0, 1)$ denote the north pole. Show (by constructing an explicit homeomorphism) that $S^2 - \{n\}$ is homeomorphic to $\mathbb{R}^2$.