Homework III: Homology and Relative Homology

due 25 Nov, 2010

A Let K be a d-dimensional simplicial complex and let $K^{(i)} = \{\sigma \in K \mid dim(\sigma) \leq i\}$ be its i-skeleton. Give an argument to demonstrate the following relationship between the homology groups $\mathsf{H}_p(K^{(i)})$ and $\mathsf{H}_p(K)$:

$$\mathsf{H}_p(K^{(i)}) = \begin{cases} \mathsf{H}_p(K) & p < i \\ 0 & p > i \\ \mathsf{Z}_p(K) & p = i \end{cases}$$

B Fix a dimension $d \geq 1$ and let $f : \mathbb{B}^d \to \mathbb{B}^d$ be a *contracting map*. This is a continuous map with the property that there exists some constant $\delta < 1$ such that the inequality $|f(x) - f(y)| \leq \delta |x - y|$ holds for all $x, y \in \mathbb{B}^d$. Prove that any such map f must have a *unique* fixed point:

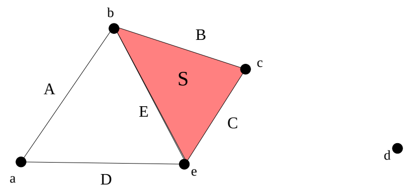


Figure 1: The vertex d is also part of K.

C Consider the simplicial complex K as drawn in Figure 1.

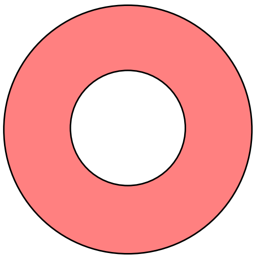
(a) For each $p = 0, 1, 2$, choose an ordering of the p-simplices in K (I suggest using alphabetical order!) and write down the boundary matrices D_p in these ordered bases:

(b) Reduce each matrix to Smith Normal Form. If you're feeling adventurous, keep track of the matrices that you use to perform the row and column operations. What are the ranks of $\mathsf{Z}_p(K)$ and $\mathsf{B}_p(K)$ for $p = 0, 1, 2$? If you can, find bases for each from your reduction:

(c) Read off the Betti numbers $\beta_0(K), \beta_1(K), \beta_2(K)$:

D Again, let K be the simplicial complex in Figure 1. Now let K_0 be the subcomplex consisting of the vertices e, b, c and the edges E, B, C (and nothing else!). Compute the relative homology groups $\mathsf{H}_p(K, K_0)$, either by matrix reduction or just mentally:

E Let $S = \{x, y, w\}$ consist of three points in the plane which form an acute triangle but are otherwise in general position. Imagine gradually increasing r while computing the Alpha-complex $Alpha_S(r)$ at each step. For $p = 0$ and $p = 1$, make a graph which plots the Betti numbers $\beta_p(Alpha_S(r))$ on the vertical axis against r on the horizontal axis:



F

Figure 2: $\mathbb{X}$ is an annulus, and $\mathbb{X}_0$ is the union of the outer and the inner circle

All of these questions concern the pair of spaces $(\mathbb{X}, \mathbb{X}_0)$ where, $\mathbb{X}$ is the annulus drawn in Figure 2, and $\mathbb{X}_0$ is the union of the outer and inner circles. Recall that we calculated the relative homology groups $\mathsf{H}_p(\mathbb{X}, \mathbb{X}_0)$ in class.

(a) Compute the absolute homology groups $\mathsf{H}_p(\mathbb{X}_0)$ and $\mathsf{H}_p(\mathbb{X})$ in every dimension:

(b) Let i denote the inclusion of $\mathbb{X}_0$ into $\mathbb{X}$ and j denote the inclusion of pairs $(\mathbb{X}, \varnothing) \to (\mathbb{X}, \mathbb{X}_0)$. Prove that $\operatorname{im} i_* = \ker j_*$:

(c) Suppose $\mathbb{Y}_0$ is just the inner circle in Figure 2. Compute $\mathsf{H}_p(\mathbb{X}, \mathbb{Y}_0)$ in every dimension: