Computational Topology: Basics

14 Oct, 2010

Today we give a brief introduction to the basic concepts of point-set topology. Our main goal is to understand what topological spaces are and when they are considered equivalent, or homeomorphic. Along the way, we will learn about several topological properties like compactness and connectedness. Next time, we will make the abstract a great deal more concrete by focusing on surfaces and classifying them up to homeomorphism.

1 Topological Spaces

Suppose that we have a set $\mathbb{X}$. A **topology** on $\mathbb{X}$ is a set $\mathcal{T}$ of subsets of $\mathbb{X}$ which satisfies the following properties:

- $\varnothing, \mathbb{X} \in \mathcal{T}$.
- Whenever $\mathcal{T}$ contains an arbitrary collection $\{U_\alpha\}$ of sets, it also contains their union: $\bigcup_\alpha U_\alpha \in \mathcal{T}$.
- Whenever $\mathcal{T}$ contains two sets U_1 and U_2, it also contains their intersection: $U_1 \cap U_2 \in \mathcal{T}$.

If a set $U \in \mathcal{T}$, we call it an *open set*, while if the complement $U^C = \mathbb{X} - U \in \mathcal{T}$, we say that U is a *closed set*. The pair $(\mathbb{X}, \mathcal{T})$ is called a **topological space**, although we often refer to $\mathbb{X}$ itself as a topological space, assuming the choice of $\mathcal{T}$ is clear from context.

Example Let's take $\mathbb{X} = \mathbb{R}$ and define $\mathcal{T}$ to be the set of all unions of open intervals, as well as the empty set. We show that $\mathcal{T}$ is indeed a topology on $\mathbb{X}$ by verifying the three above properties in turn:

- $\varnothing \in \mathcal{T}$ by definition, while $\mathbb{R} = \bigcup_n (-n, n) \in \mathcal{T}$, as well.
- If you take the union of two unions of open intervals, then you get a union of open intervals.
- The intersection of two open intervals is either another open interval or the empty set.

This topology is often called the *usual* or *Euclidean* topology on $\mathbb{R}$.

Note that in general the intersection of a *finite* number of open sets will also be open (by induction). However this need no longer be true for the intersection of an *infinite* number of open sets. To see this, note that $\{0\}$ is the intersection of the open sets $(-\frac{1}{n}, \frac{1}{n})$, but that $\{0\}$ cannot be an open set.

Basis To make definitions and arguments simpler, we need one more concept. Given a topology $\mathcal{T}$ on $\mathbb{X}$, a **basis** $\mathcal{B}$ for $\mathcal{T}$ is a collection of sets in $\mathcal{T}$ such that every element in $\mathcal{T}$ can be written as a union of elements of $\mathcal{B}$. In this case, we say that $\mathcal{B}$ generates $\mathcal{T}$. In the example above, a possible choice of basis would be the set of all open intervals, plus the empty set.

More examples Let $\mathbb{X} = \mathbb{R}$ and let $\mathcal{T}$ be generated by the collection of half-open intervals $[a, b)$, for all $a < b \in \mathbb{R}$. Then this is also a topological space, called the *half-open topology*.

Let $\mathbb{X}$ be an infinite set and define $\mathcal{T} = \{U \subseteq \mathbb{X} \mid |U^C| < \infty\}$, plus the empty set. Then this *is* a topology, and is usually called the co-finite topology on $\mathbb{X}$. We verify this:

We take $\mathbb{X} = \mathbb{R}$ with the usual topology $\mathcal{T}$ and we consider $\mathbb{Y} = [0, 1)$. Then the collection of sets $\{\mathbb{Y} \cap U \mid U \in \mathcal{T}\}$ is a topology on $\mathbb{Y}$. This construction works in general, and creates the *subspace* topology on a subset of a topological space.

Given two topological spaces $(\mathbb{X}, \mathcal{S})$ and $(\mathbb{Y}, \mathcal{T})$, we put a topology on $\mathbb{X} \times \mathbb{Y} = \{(x, y) \mid x \in \mathbb{X}, y \in \mathbb{Y}\}$ by taking $\mathcal{S} \times \mathcal{T}$ as a basis. In this way we create the usual topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and so forth.

2 Continuity

Suppose that $(\mathbb{X}, \mathcal{T})$ and $(\mathbb{X}', \mathcal{T}')$ are two topological spaces. A function $f : \mathbb{X} \to \mathbb{X}'$ is called **continuous** if $U' \in \mathcal{T}'$ implies that $f^{-1}(U') \in \mathcal{T}$; in other words, if the inverse image of every open set is an open set. Note that this generalizes the familiar "$\epsilon - \delta$" definition of a continuous function from $\mathbb{R} \to \mathbb{R}$. It is also easy to see that the composition of two continuous functions is again a continuous function:

Homeomorphisms First we set some terminology about functions. A set function $f : \mathbb{X} \to \mathbb{Y}$ is called **injective** if $f(x) = f(y)$ implies $x = y$, and it is called **surjective** if, for every $y \in \mathbb{Y}$, there exists an $x \in \mathbb{X}$ such that $f(x) = y$. If f is both injective and surjective, we call it a **bijection**; in this case, f has a unique inverse $f^{-1} : \mathbb{Y} \to \mathbb{X}$.

A **homeomorphism** between two topological spaces $\mathbb{X}$ and $\mathbb{Y}$ is a continuous bijection which has a continuous inverse. Example: the unit circle and the unit square, both with the induced topology from $\mathbb{R}^2$, are homeomorphic:

Homotopies A **homotopy** between two topological spaces $\mathbb{X}$ and $\mathbb{Y}$ is a function $F : \mathbb{X} \times I \to \mathbb{Y}$, where $I = [0, 1]$ with the induced topology. Two functions $f, h : \mathbb{X} \to \mathbb{Y}$ are **homotopic** ($f \simeq g$) if there exists a homotopy F such that, for every $x \in \mathbb{X}$, $F(x, 0) = f(x)$ and $F(x, 1) = h(x)$. Finally, a function $f : \mathbb{X} \to \mathbb{Y}$ is a **homotopy equivalence** if there exists a function $g : \mathbb{Y} \to \mathbb{X}$ such that $g \circ f \simeq id_{\mathbb{X}}$ and $f \circ g \simeq id_{\mathbb{Y}}$.

For example, the unit circle S^1, and the punctured plane $\mathbb{R}^2 - \{0\}$ are homotopically equivalent but not homeomorphic. To see the former claim, use radial projection. For the latter, keep reading.

3 In-Class Problems: playing around with definitions

A A topological space $(\mathbb{X}, \mathcal{T})$ is called **Hausdorff** if: for every pair of distinct points $x, y \in \mathbb{X}$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$. Which of the following spaces are Hausdorff?

(a) $\mathbb{R}$ with the usual topology:

(b) $\mathbb{R}$ with the cofinite topology:

B The *discrete* topology on a set $\mathbb{X}$ declares every single subset of $\mathbb{X}$ to be open. The *indiscrete* topology on $\mathbb{X}$ has only two open subsets: $\varnothing$ and $\mathbb{X}$ itself.

(a) Describe all continuous functions $f : \mathbb{X} \to \mathbb{R}$, assuming that $\mathbb{X}$ has the discrete topology and $\mathbb{R}$ has the usual topology:

(b) Describe all continuous functions $f : \mathbb{X} \to \mathbb{R}$, assuming that $\mathbb{X}$ has the indiscrete topology and $\mathbb{R}$ has the usual topology

C Construct a homotopy between any two continuous functions $f : \mathbb{X} \to \mathbb{R}$, where $\mathbb{X}$ is any topological space and $\mathbb{R}$ has the usual topology:

4 Some Topological Properties

In order to show that two topological spaces $\mathbb{X}$ and $\mathbb{Y}$ are homeomorphic, one need only construct a homeomorphism between them. To show they are *not* homeomorphic is trickier. Since we can't consider and then reject every possible function in turn, we instead need to find a *topological property* which one has and the other does not. In a nutshell, a topological property is something one can say about a space in terms of open sets. Alternatively, it is a property which is preserved by any homeomorphism.

Connectedness A topological space $\mathbb{X}$ is called *disconnected* if $\mathbb{X}$ can be written as the union of two disjoint, non-empty, subsets. Otherwise, we say that $\mathbb{X}$ is *connected*.

Proposition: A topological space $\mathbb{X}$ is disconnected iff $\mathbb{X}$ contains a non-empty proper subset which is both open and closed:

For example, $\mathbb{R}$ with the usual topology is connected, while $\mathbb{R}$ with the half-open topology is disconnected:

Proposition: If $f : \mathbb{X} \to \mathbb{Y}$ is continuous, and $\mathbb{X}$ is connected, then $f(\mathbb{X})$ is also connected.

Proof:

As a consequence, we see that connectedness is a topological property: a connected space cannot be homeomorphic to a disconnected space. We can also use this idea to see that a line segment is not homeomorphic to the unit circle, both with the induced topology from $\mathbb{R}^2$:

Compactness Suppose that $\mathbb{X}$ is a topological space and $\mathbb{Y}$ is a subspace. A collection of open set $\{U_\alpha\}$ is called an *open cover* of $\mathbb{Y}$ if $\mathbb{Y} \subseteq \bigcup_\alpha U_\alpha$. $\mathbb{Y}$ is called **compact** if every open cover can be reduced to a *finite* sub-cover: in other words, if there always exists some finite subcollection $U_1, \ldots, U_n \in \{U_\alpha\}$ such that $\mathbb{Y} \subseteq U_1 \cup \ldots \cup U_n$.

This definition is quite technical. For more intuition, we can appeal to the following fact: if $\mathbb{Y}$ is a subspace of $\mathbb{R}^N$, and has the topology inherited from the usual topology, then $\mathbb{Y}$ is compact iff it is closed and bounded:

Proposition: Let $f : \mathbb{X} \to \mathbb{Y}$ be a continuous function and suppose that $\mathbb{X}$ is compact. Then $f(\mathbb{X})$ is also compact.

As promised, we see that S^1 and $\mathbb{R}^2 - \{0\}$ are not homeomorphic, since the first space is compact and the second one is not.

5 More Exercises

A Which of the following spaces are compact:

(a) $\mathbb{R}$ with the usual topology:

(b) $\mathbb{R}$ with the co-finite topology:

B A path between two points x and y in a space $\mathbb{X}$ is a continuous function $f : [0, 1] \rightarrow \mathbb{X}$ such that $f(0) = x$ and $f(1) = y$. The space $\mathbb{X}$ is called path-connected if there exists a path between any two points in the space. Show that every path-connected space is connected (note: the converse is false, although this is tricky):