Exact Sequences

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1 Overview

Last time we discussed *functoriality*: how continuous maps between spaces (or pairs of spaces) help transfer homological (or relative homological) information. Today we will see that these maps allow us to see connections between:

- the relative homology of a pair of spaces $(\mathbb{X}, \mathbb{A})$ and the absolute homology of the two spaces $\mathbb{X}$ and $\mathbb{A}$.
- the homology of a decomposed space $\mathbb{X} = \mathbb{X}' \cup \mathbb{X}''$ in terms of the homology of the spaces $\mathbb{X}'$, $\mathbb{X}''$, and $\mathbb{X}' \cap \mathbb{X}''$.

The second idea in particular is very important, for it will allow us to understand the homology of complicated spaces via a glueing-together of simpler spaces. For both ideas, the key concept is that all of the relevant homology groups fit together into a long exact sequence; we begin with an abstract description of this algebraic concept, before moving on to more concrete examples.

2 Algebraic Terminology

Let's briefly review some terminology about maps between vector spaces before we get to the new idea. Given a homomorphism $f : \mathsf{V} \to \mathsf{W}$, recall that we defined the *kernel* of f to be $\ker f = \{v \in \mathsf{V} \mid f(v) = 0\}$, and the *image* of f to be $\operatorname{im} f = \{f(v) \mid v \in \mathsf{V}\}$. We also define the *cokernel* of f to be the quotient $\operatorname{cok} f = \mathsf{W}/\operatorname{im} f$. Note two convenient facts: $\mathsf{V} \cong \ker f \oplus \operatorname{im} f$ and $\mathsf{W} \cong \operatorname{im} f \oplus \operatorname{cok} f$.

Now suppose we have a sequence of three vector spaces connected by maps:

$$\mathsf{V} \xrightarrow{f} \mathsf{W} \xrightarrow{g} \mathsf{U}$$

This sequence is said to be *exact* at W if $\operatorname{im} f = \ker g$; that is, if an element in W is mapped to zero by g iff it is the image of an element under f.

To relate exactness to earlier ideas, consider the following obvious facts:

- A map $f : \mathsf{V} \to \mathsf{W}$ is injective iff the sequence

  $$0 \longrightarrow \mathsf{V} \xrightarrow{f} \mathsf{W}$$

  is exact at V.

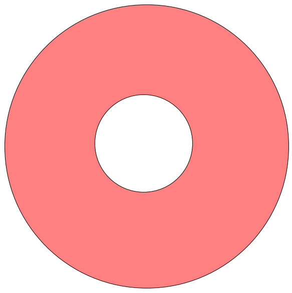

Figure 1: $\mathbb{X}$ is the annulus, and $\mathbb{A}$ the union of the inner and the outer circles

- A map $g : \mathsf{W} \to \mathsf{U}$ is surjective iff the sequence

$$\mathsf{W} \xrightarrow{g} \mathsf{U} \longrightarrow 0$$

  is exact at U.

A *short exact sequence* is a sequence of five vector spaces, bracketed by zero, which is exact at all three internal nodes:

$$0 \longrightarrow \mathsf{V} \xrightarrow{f} \mathsf{W} \xrightarrow{g} \mathsf{U} \longrightarrow 0$$

Finally, a *long exact sequence* is a doubly infinite sequence of vector spaces which is exact at every node.

3 Long Exact Sequence of a Pair

Suppose we have a pair of spaces $(\mathbb{X}, \mathbb{A})$. We will now see that the absolute homology of the two individual spaces $\mathbb{X}$ and $\mathbb{A}$ fits into a long exact sequence with the relative homology of the pair $(\mathbb{X}, \mathbb{A})$. Before giving the general theorem, let us work through a familiar example.

3.1 Annulus example.

Consider again the annulus $\mathbb{X}$ in Figure 1, where $\mathbb{A}$ is the union of the inner and the outer circles. Let $i : \mathbb{A} \to \mathbb{X}$ be the inclusion map which considers $\mathbb{A}$ as a subspace of $\mathbb{X}$, and let $j : \mathbb{X} \to (\mathbb{X}, \mathbb{A})$ be the map of pairs. We claim that $\operatorname{im} i_* = \ker j_*$ in each dimension. Let's investigate this in the two relevant dimensions.

Dimension one. The space $\mathbb{A}$ is the disjoint union of two circles, so clearly $\mathsf{H}_1(\mathbb{A})$ is rank two; let's choose as a basis the homology classes $[A]$ and $[B]$, where A and B are 1-cycles made up of the inner and outer circles, respectively. On the other hand, the group $\mathsf{H}_1(\mathbb{X})$ is rank one (since the annulus is homotopically equivalent to a single circle!); as generator, we take the homology class $[C]$, where C is some 1-cycle going around the hole in the annulus.

Note that A and B are both homologous to C *within* $\mathbb{X}$; in other words, $[C] = i_*([A]) = i_*([B])$, and so in this case $\operatorname{im} i_* = \mathsf{H}_1(\mathbb{X})$. On the other hand, we also have $j_*([C]) = 0$ since the homology between C and A (a chain fully within the space $\mathbb{A}$!) means that $[C]$ becomes trivial once we mod out by the space $\mathbb{A}$. In other words, we have shown that the sequence

$$\mathsf{H}_1(\mathbb{A}) \xrightarrow{i_*} \mathsf{H}_1(\mathbb{X}) \xrightarrow{j_*} \mathsf{H}_1(\mathbb{X}, \mathbb{A})$$

is exact at the middle node. Notice also that i_* is not injective; indeed, $\ker i_*$ is rank one, generated by $[A] + [B]$, a fact we will come back to later.

Dimension zero. Since $\mathbb{A}$ has two connected components, the group $\mathsf{H}_0(\mathbb{A})$ is rank two, and let's take as generators the classes $[v]$ and $[w]$ where v is some vertex on the inner circle and w is some vertex on the outer circle. The annulus is connected and hence $\mathsf{H}_0(\mathbb{X})$ is rank one, with generator $[x]$. Note that $[x] = i_*([v]) = i_*([w])$, and hence $[x] \in \operatorname{im} i_*$. On the other hand, $j_*([x]) = 0$, since $\mathsf{H}_0(\mathbb{X}, \mathbb{A})$ is the zero group. As above, we have shown that the sequence

$$\mathsf{H}_0(\mathbb{A}) \xrightarrow{i_*} \mathsf{H}_0(\mathbb{X}) \xrightarrow{j_*} \mathsf{H}_0(\mathbb{X}, \mathbb{A})$$

is exact at the middle.node

Connecting homomorphisms We now stitch the last two sequences together via a *connecting homomorphism*, as we now describe. Recall from last week that the relative homology group $\mathsf{H}_1(\mathbb{X}, \mathbb{A})$ is rank one, with generator $[E]$, where E is a one-chain which connects some point on the inner circle to some point on the outer circle. We define a homomorphism $\partial_* : \mathsf{H}_1(\mathbb{X}, \mathbb{A}) \to \mathsf{H}_0(\mathbb{A})$ by setting $\partial_*([E]) = [\partial E] \in \mathsf{H}_0(\mathbb{A})$; in other words, we map $[E]$ to the homology class represented by its boundary in $\mathbb{A}$.

Of course, we could also write $\partial_*([E]) = [v] + [w]$, since any vertex on the inner circle (outer circle) is homologous to v (w). On the other hand, we see from the above that $i_*([v] + [w]) = 0$, and hence $\operatorname{im} \partial_* = \ker i_*$. That is, we have extended the above exact sequence into this one:

$$\mathsf{H}_1(\mathbb{A}) \xrightarrow{i_*} \mathsf{H}_1(\mathbb{X}) \xrightarrow{j_*} \mathsf{H}_1(\mathbb{X}, \mathbb{A}) \xrightarrow{\partial_*} \mathsf{H}_0(\mathbb{A}) \xrightarrow{i_*} \mathsf{H}_0(\mathbb{X}) \xrightarrow{j_*} \mathsf{H}_0(\mathbb{X}, \mathbb{A})$$

Really, we also have to show exactness at the third node. But this is easy. Note that $\ker \partial_* = 0$, since the group $\mathsf{H}_1(\mathbb{X}, \mathbb{A})$ is rank one, and the generator does not map to zero. On the other hand, the image of $\mathsf{H}_1(\mathbb{X})$ under j_* is also zero, since the generator of the rank one absolute homology group does map to zero; hence $\operatorname{im} j_* = 0 = \ker \partial_*$.

Following a similar procedure, we can extend the sequence yet further. Consider the relative homology group $\mathsf{H}_2(\mathbb{X}, \mathbb{A})$, which is rank-one, and take as generator the homology class $[\Gamma]$, where Γ is a 2-chain making up the entire annulus. Exactly as above, we define a connecting homomorphism $\partial_* : \mathsf{H}_2(\mathbb{X}, \mathbb{A}) \to \mathsf{H}_1(\mathbb{A})$ by $\partial_*([\Gamma]) = [\partial\Gamma] = [A] + [B]$ and we see immediately that $\operatorname{im} \partial_* = \ker i_*$. Thus, we can extend the sequence yet further (now we show only the leftmost part):

$$\mathsf{H}_2(\mathbb{A}) \xrightarrow{\quad i_* \quad} \mathsf{H}_2(\mathbb{X}) \xrightarrow{\ j_* \ } \mathsf{H}_2(\mathbb{X}, \mathbb{A}) \xrightarrow{\ \partial_* \ } \mathsf{H}_1(\mathbb{A}) \xrightarrow{\quad i_* \quad} \mathsf{H}_1(\mathbb{X}) \xrightarrow{\ j_* \ } \mathsf{H}_1(\mathbb{X}, \mathbb{A})$$

Note that exactness at the second node in this sequence is trivial, since the first two groups in this sequence are zero, while exactness at the third node follows from ∂_* being injective in this dimension. Since all absolute and homology groups above dimension two (and in negative dimension) are zero for this example, we have in fact produced an entire long exact sequence, which we call the long exact sequence of the pair $(\mathbb{X}, \mathbb{A})$

3.2 General Theory and an Application

There was nothing special hidden in the previous example. Given any pair of spaces $(\mathbb{X}, \mathbb{A})$, we can define for each integer p a connecting homomorphism $\partial_* : \mathsf{H}_p(\mathbb{X}) \to \mathsf{H}_{p-1}(\mathbb{A})$ in similar fashion to above. Suppose we have a homology class α in $\mathsf{H}_p(\mathbb{X}, \mathbb{A})$ with relative p-cycle representative A. Since A is a relative p-cycle, there are only two possibilities: either A has a completely empty boundary, or the boundary of A lies entirely within $\mathbb{A}$. In either case, we define $\partial_*(\alpha) \in \mathsf{H}_{p-1}(\mathbb{A})$ to be the homology class of the $(p-1)$-cycle ∂A. Notice that $\partial_*(\alpha) = 0$ iff A is also an *absolute* p-cycle, in which case α is of course in the image of j_*; that is, $\ker \partial_* = \operatorname{im} j_*$.

Moving through all the dimensions p, we can stitch the connecting homomorphisms together with the maps i_* and j_* to form the *long exact sequence of the pair* $(\mathbb{X}, \mathbb{A})$:

$$\cdots \xrightarrow{\quad \partial_* \quad} \mathsf{H}_p(\mathbb{A}) \xrightarrow{\quad i_* \quad} \mathsf{H}_p(\mathbb{X}) \xrightarrow{\quad j_* \quad} \mathsf{H}_p(\mathbb{X}, \mathbb{A}) \xrightarrow{\ \partial_* \ } \mathsf{H}_{p-1}(\mathbb{A}) \xrightarrow{\quad i_* \quad} \cdots$$

This sequence is often very computationally useful. In particular, if we know every two out of three of the groups in the sequence, then exactness permits us to figure out the other ones. For example:

CLAIM: For all $d \geq 1$, the relative homology group $\mathsf{H}_p(\mathbb{B}^d, \mathbb{S}^{d-1})$ is rank one when $p = d$ and is zero when $p \neq d$:

proof:

4 The Snake Lemma

Let us set topology aside for a second and take a brief trip into a more purely algebraic world. We will show that the process above abstracts into an extremely useful one: namely, whenever we have a short exact sequence of chain complexes, we can produce a long exact sequence of homology groups. The topological payoff will come in the next section, where we learn that we can understand the homology of a large space from the homology of smaller spaces which glue together to form it, as long as we also understand the glueing itself. First some definitions.

Chain complexes and chain maps. A *chain complex* is a sequence of vector spaces U_p, one for each integer p, along with a sequence of homomorphisms $u_p : \mathsf{U}_p \to \mathsf{U}_{p-1}$ which satisfy the "square-zero" property: $u_p \circ u_{p+1} = 0$. We usually think of $\mathcal{U} = (\mathsf{U}_p, u_p)$ as the chain complex and we call the maps u_p boundary maps. Exactly as before, we can define cycle groups $\mathsf{Z}_p(\mathcal{U}) = \ker u_p$ and boundary groups $\mathsf{B}_p(\mathcal{U}) = \operatorname{im} u_{p+1}$. The square-zero property implies that $\mathsf{B}_p(\mathcal{U}) \subseteq \mathsf{Z}_p(\mathcal{U})$, and thus we can define $\mathsf{H}_p(\mathcal{U}) = \mathsf{Z}_p(\mathcal{U})/\mathsf{B}_p(\mathcal{U})$, calling this the p-th homology group of the chain complex $\mathcal{U}$.

Now let $\mathcal{V} = (\mathsf{V}_p, v_p)$ be another chain complex. A *chain map* $\mathcal{U} \to \mathcal{V}$ between $\mathcal{U}$ and $\mathcal{V}$ is a sequence of homomorphisms $\phi_p : \mathsf{U}_p \to \mathsf{V}_p$, one for each integer p, which must commute with the boundary maps: in other words, we must have the equation $v_p \circ \phi_p = \phi_{p-1} \circ u_p$ for every p (Note that we've already seen an example of this: the maps $f_\# : \mathsf{C}_p(K) \to \mathsf{C}_p(L)$ induced by a simplicial map make up a chain map between simplicial chain complexes.). Since the chain map commutes with the boundary map, we know that cycles go to cycles and boundaries go to boundaries: $\phi_p(\mathsf{Z}_p(\mathcal{U})) \subseteq (\mathsf{Z}_p(\mathcal{V})$ and $\phi_p(\mathsf{B}_p(\mathcal{U})) \subseteq \mathsf{B}_p(\mathcal{V})$. Hence the chain map induces a map on homology groups in every dimension, which we denote $(\phi_p)_* : \mathsf{H}_p(\mathcal{U}) \to \mathsf{H}_p(\mathcal{V})$.

Let $\mathcal{W} = (\mathsf{W}_p, w_p)$ be a third chain complex and suppose we have another chain map $\mathcal{V} \to \mathcal{W}$ made up of the homomorphisms $\psi_p : \mathsf{V}_p \to \mathsf{W}$. We say that the sequence $\mathcal{U} \to \mathcal{V} \to \mathcal{W}$ is *exact* at $\mathcal{V}$ if $\ker \psi_p = \operatorname{im} \phi_p$ for every p. A *short exact sequence of chain complexes* is an exact sequence like the one below:

$$0 \longrightarrow \mathcal{U} \xrightarrow{\ \phi\ } \mathcal{V} \xrightarrow{\ \psi\ } \mathcal{W} \longrightarrow 0$$

Notice that this is really a compact way to represent a great deal of algebraic information. In truth, a short exact sequence of chain complexes is a grid with five columns and infinitely many rows (or you can transpose the thing if you like!). Each column stores a chain complex written vertically, with boundary maps going from top to bottom, and each row stores an exact sequence of vector spaces for a fixed dimension p. The fact that we have chain maps means that every single square in the diagram commutes:

For example, consider a pair of subcomplexes (K, K_0). Then the inclusions of K_0 into K and K into (K, K_0) induce the following short exact sequence of chain complexes:

$$0 \longrightarrow \mathcal{C}(K_0) \longrightarrow \mathcal{C}(K) \longrightarrow \mathcal{C}(K, K_0) \longrightarrow 0$$

Recall above that we were able to construct a connecting homomorphism $\partial_* : \mathsf{H}_p(K, K_0) \to \mathsf{H}_{p-1}(K_0)$ that then produced a long exact sequence of homology groups. This idea generalizes, as we now discuss.

Connecting homomorphisms and the snake lemma. Assume that we have some short exact sequence of chain complexes as in Diagram 4. Now we "pass to homology" and see what happens. Since we have chain maps ϕ and ψ, they induce for every p a sequence of homology maps

$$\mathsf{H}_p(\mathcal{U}) \xrightarrow{(\phi_p)_*} \mathsf{H}_p(\mathcal{V}) \xrightarrow{(\psi_p)_*} \mathsf{H}_p(\mathcal{W})$$

For each p, we can stitch together the p-th such sequence to the $(p-1)$-th sequence via a connecting homomorphism $D_p : \mathsf{H}_p(\mathcal{W}) \to \mathsf{H}_{p-1}(\mathcal{U})$, whose definition we now briefly describe:

SNAKE LEMMA: Given the short exact sequence of chain complexes in Diagram 4, passing to homology along with the induced maps and the connecting homomorphisms produces a long exact sequence of homology groups:

$$\cdots \xrightarrow{D_{p+1}} \mathsf{H}_p(\mathcal{U}) \xrightarrow{(\phi_p)_*} \mathsf{H}_p(\mathcal{V}) \xrightarrow{(\psi_p)*} \mathsf{H}_p(\mathcal{W}) \xrightarrow{D_p} \mathsf{H}_{p-1}(\mathcal{U}) \xrightarrow{(\phi_p)_*} \cdots$$

5 Mayer-Vietoris Exact Sequence

We now exploit the Snake Lemma to describe the homology of two spaces in terms of the homology of their union and the homology of their intersection. To avoid theoretical complication, we discuss this only on the simplicial level, but it can also be done for arbitrary topological spaces and singular homology.

Decomposition. Suppose we have a simplicial complex K, along with two subcomplexes K', K" such that $K' \cup K'' = K$. Of course, their intersection $L = K' \cap K''$ is also a subcomplex. We consider three chain complexes. First, the simplicial chain complexes $\mathcal{C}(L) = (\mathsf{C}_p(L), \partial_p^L)$ and $\mathcal{C}(K) = (\mathsf{C}_p(K), \partial_p^K)$; note that the boundary map on L is of course just the restriction to L of the boundary map on K. We also form the direct sum chain complex $\mathcal{C}(K') \oplus \mathcal{C}(K'') = (\mathsf{C}_p(K') \oplus \mathsf{C}_p(K''), \partial_p^{K'} \oplus \partial_p^{K''})$, where the boundary map is defined component-wise. Note that this is just the chain complex of the *disjoint* union of K' and K'', where we just mentally ignore the fact that they might have non-empty intersection, and that the homology of the direct sum chain complex is the direct sum of the respective homology groups.

Exact sequence. These three chain complexes fit into a short exact sequence, as we now describe. First, we let i' and i'' denote the inclusions of $\mathcal{C}(L)$ into, respectively, $\mathcal{C}(K')$ and $\mathcal{C}(K'')$, and we let j' and j'' denote the inclusions of $\mathcal{C}(K')$ and $\mathcal{C}(K'')$, respectively, into $\mathcal{C}(K)$. Finally, it is easy to see that the sequence of chain complexes

$$0 \longrightarrow \mathcal{C}(L) \xrightarrow{i} \mathcal{C}(K') \oplus \mathcal{C}(K'') \xrightarrow{j} \mathcal{C}(K) \longrightarrow 0$$

where $i = i' \oplus i''$ and $j = j' + j''$, is exact. Appealing to the Snake Lemma, we obtain the following essential result:

MAYER-VIETORIS SEQUENCE THEOREM: Let K be a simplicial complex decomposed into subcomplexes $K = K' \cup K''$ and set $L = K' \cap K''$. Then there exists a long exact sequence of simplicial homology groups:

$$\cdots \xrightarrow{D} \mathsf{H}_p(L) \xrightarrow{i_*} \mathsf{H}_p(K') \oplus \mathsf{H}_p(K'') \xrightarrow{j_*} \mathsf{H}_p(K) \xrightarrow{D} \mathsf{H}_{p-1}(L) \xrightarrow{i_*} \cdots$$

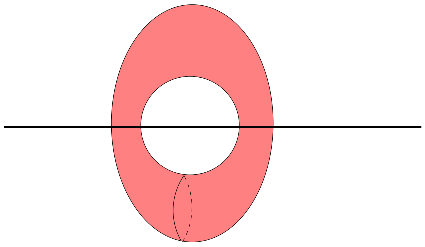

Figure 2: The torus $\mathbb{X}$ is decomposed into $\mathbb{X}'$, which lies above the black line, and $\mathbb{X}''$, which lies below.

Example To understand the Mayer-Vietoris sequence, in particular the mysterious connecting homomorphism, let us decompose the now-familiar torus as shown in Figure 2. We let $\mathbb{X}$ be the torus, and write it as $\mathbb{X} = \mathbb{X}' \cup \mathbb{X}''$, where $\mathbb{X}'$ ($\mathbb{X}''$) is the half of the torus which lies either on or above (on or below) the thick black line. We note that $\mathbb{X}'$ and $\mathbb{X}''$ are both homeomorphic to cylinders, while $\mathbb{Y} = \mathbb{X}' \cap \mathbb{X}''$ is homeomorphic to a disjoint pair of circles: